On Gleason Spaces

by

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A Thesis Presented to the

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To my father and mother

TABLE OF CONTENTS

	Page
ACKNOWLEDGEMENT	i
ABSTRACT	ii
CHAPTER 1: INTRODUCTION	1
LATTICES AND BOOLEAN ALGEBRAS	1
IDEALS AND FILTERS OF A BOOLEAN ALGEBRA	7
FIELDS OF SETS	10
COMPLETELY REGULAR SPACES	13
CONVERGENCE OF z-FILTERS	14
ZERO-DIMENSIONAL SPACES	17
CHAPTER 2: THE STONE-ČECH COMPACTIFICATION	21
DEVELOPMENT OF THE STONE-ČECH COMPACTIFICATION	21
PROPERTIES OF THE STONE-ČECH COMPACTIFICATION	25
CHARACTERIZATIONS OF PSEUDOCOMPACT SPACES	30
GLICKSBERG'S THEOREM ON THE PRODUCT OF TWO SPACES	36
CHAPTER 3: BOOLEAN ALGEBRA, QUOTIENT EXTENSION RINGS	39
THE STONE REPRESENTATION THEOREM	41
MAXIMAL RING OF QUOTIENTS	43
THE BOOLEAN ALGEBRA OF ANNIHILATOR IDEALS	47
THE BOOLEAN ALGEBRA OF IDEMPOTENTS	50
MAXIMAL IDEAL SPACES	52

	Page
CHAPTER 4: PROJECTIVE RESOLUTIONS	55
FREE OBJECTS AND PROJECTIVE OBJECTS	56
EXTREMALLY DISCONNECTED SPACES	58
EXISTENCE AND UNIQUENESS OF PROJECTIVE RESOLUTIONS	60
MAXIMAL IDEAL SPACE OF THE MAXIMAL RING OF QUOTIENTS	66
REFERENCES	68

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ABSTRACT

The object of prime interest in this thesis is the projective resolutions in the category of compact Hausdorff spaces and continuous maps. However, related topics are also discussed both for their connection with the subject and for their intrinsic interest.

Chapter one serves as an introduction that contains most of the fundamental concepts and techniques to be used in subsequent chapters. Four main topics are discussed: lattices and Boolean algebras, completely regular spaces, convergence of z-filters and z-ultrafilters, and zero-dimensional spaces. This material appears in the works of Dwinger [7], Gillman and Jerison [8], Halmos [12], and Walker [19].

Chapter two contains the development of the Stone-Čech compactification and some of its basic properties. Some characterizations of pseudocompact spaces are discussed. These characterizations culminate in Glicksberg's theorem on the Stone-Čech compactification of a product of two spaces. The discussion in this chapter is based on that of Bagley, Connell and Mcknight [1], Gillman and Jerison [8], Clicksberg [9, 10], and Hewitt [13].

The necessary material on Boolean algebras, including the Stone representation theorem, is discussed in Chapter three. The maximal ring of quotients of a commutative semisimple ring is given self-contained treatment and proves to be useful in connection with maximal

ideal spaces and projective resolutions. Parts of this material are treated in the works of Brainerd and Lambek [4], Doctor [5], Lambek [15], Utumi [18], and Walker [19].

Chapter four places major emphasis on the existence, uniqueness, and characterization of projective resolutions. This topic relates the Stone-Čech compactification to both Boolean algebras and the class of extremally disconnected spaces. Theorem 3 in this chapter is a new result on the product of Gleason spaces while Corollary 2 provides a new simple proof of the fact that a projective object is extremally disconnected. The discussion in this chapter is based on the works of Banaschewski [2, 3], Hager [11], Park [16], and Rainwater [17].

CHAPTER 1

INTRODUCTION

A partially ordered set is a set X together with a binary relation $\leq$ defined which is reflexive, transitive and antisymmetric (i.e. $x \leq y$ and $y \leq x$ implies $x = y$). Two elements x and y of X are comparable if either $x \leq y$ or $y \leq x$. A partially ordered set is a chain (also totally or linearly ordered set) if any two of its elements are comparable. The set of all subsets of a set ordered by set inclusion is a partially ordered set but not a chain.

Lattices: An upper bound for a subset A of a partially ordered set X is an element x such that $a \leq x$ for all $a \in A$, and a lower bound for A is an element z such that $z \leq a$ for all $a \in A$. The least upper bound (l.u.b) for a family $\{x_\alpha\}$ of members of X is denoted by $V_\alpha x_\alpha$ and the greatest lower bound (g.l.b) is written $\Lambda_\alpha x_\alpha$. The terms join and supremum (respectively meet and infimum) are among the several widely adopted names of least upper bound (respectively greatest lower bound).

A lattice is a partially ordered set in which every pair of elements has a supremum and an infimum. A lattice is said to be complete if every subset of the lattice has a supremum and an infimum.

Examples of Lattices

(1) The set of all subsets of a set. Infimum and supremum are here the usual set-theoretic operations of intersection and union.

(2) The lattice of all open (closed) subsets of a topological space.

(3) The ring $C(X)$ of all continuous real-valued functions on the topological space X is a lattice with the partial order defined by $f \leq g$ if $f(x) \leq g(x)$ for every point x of X. $(f \vee g)(x) = \max\{ f(x), g(x) \}$ and $(f \wedge g)(x) = \min\{ f(x), g(x) \}$.

Boolean Algebras: A lattice is called distributive if the operations meet and join satisfy the two identities:

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

and

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$

The two identities can be shown to be equivalent. A lattice is called complemented if it contains distinct elements 0 and 1 such that $0 \leq x \leq 1$ for all elements x and to each element x is assigned an element x' such that $x \vee x' = 1$ and $x \wedge x' = 0$. The element x' is called the complement of x.

A Boolean algebra is a complemented distributive lattice. A Boolean algebra is said to be complete if it is complete as a lattice.

Examples of Boolean Algebras

(1) The family of all subsets of a set is a Boolean algebra and this example is complete.

(2) The family of all regular closed subsets of a topological space is also a complete Boolean algebra. A subset F of a space X is called a regular closed set if $F = cl(int\ F)$. Similarly, a subset G of X is said to be a regular open set if $G = int(cl\ G)$.

Theorem 1. The family $R(X)$ of regular closed subsets of a space X is a complete Boolean algebra with the following operations:

(1) $A \leq B$ if and only if $A \subseteq B$

(2) $\bigvee_\alpha A_\alpha = cl(\bigcup_\alpha int\ A_\alpha)$.

(3) $\bigwedge_\alpha A_\alpha = cl(int \bigcap_\alpha A_\alpha)$.

(4) $A' = cl(X \setminus A)$.

Proof: Let $\{A_\alpha\}$ be any collection of regular closed subsets of X. Write $M = cl(\bigcup_\alpha int\ A_\alpha)$. Then M is the closure of an open set and is clearly in $R(X)$. Since M contains $cl(int\ A_\alpha) = A_\alpha$

for each α, M is an upper bound of $\{A_\alpha\}$. Suppose that N is another upper bound. Then $\operatorname{int} A_\alpha$ is contained in $\operatorname{int} N$ for every α and therefore

$$M = \operatorname{cl}(\textstyle\bigcup_\alpha \operatorname{int} A_\alpha) \subseteq \operatorname{cl}(\operatorname{int} N) = N.$$

Hence, M is the least upper bound of $\{A_\alpha\}$.

Let $L = \operatorname{cl}(\operatorname{int}(\bigcap_\alpha A_\alpha))$. Since $\operatorname{int}(\bigcap_\alpha A_\alpha)$ is contained in $\operatorname{int} A_\alpha$ for all α, L is contained in A_α for every α and L is a lower bound for $\{A_\alpha\}$. Now suppose that N is another lower bound. Then N is contained in $\bigcap A_\alpha$ so that $\operatorname{int} N$ is contained in $\operatorname{int}(\bigcap_\alpha A_\alpha)$. Therefore,

$$N = \operatorname{cl}(\operatorname{int} N) \subseteq \operatorname{cl}(\operatorname{int}(\textstyle\bigcap_\alpha A_\alpha)) = L$$

and L is the greatest lower bound.

Take the empty set and X to be the zero element and the unit element, respectively. For any A in $R(X)$, write $A' = \operatorname{cl}(X \setminus A)$. Then $A \wedge A'$ is the closure of the interior of the boundary of A and is therefore empty. Similarly, $A \vee A'$ is the closure of a dense set and is therefore all of X.

We still need to show that the distributive law holds. Let A, C, and D belong to $R(X)$. Note that

$$\mathrm{cl}(\mathrm{int}\, C \cup \mathrm{int}\, D) = C \vee D = \mathrm{cl}(\mathrm{int}(C \vee D)).$$

Now recall that in any topological space, if two sets S and T have the same closure and G is an open set, then

$$\mathrm{cl}(G \cap S) = \mathrm{cl}(G \cap T).$$

Applying these two formulas, we obtain the distributive law:

$$\begin{aligned}
A \wedge (C \vee D) &= \mathrm{cl}[\,\mathrm{int}[A \cap (C \vee D)]\,] \\
&= \mathrm{cl}(\,\mathrm{int}\, A \cap \mathrm{int}(C \vee D)), \quad \text{since } \mathrm{int}(S \cap T) = \mathrm{int}\, S \cap \mathrm{int}\, T \\
&= \mathrm{cl}(\,\mathrm{int}\, A \cap (\,\mathrm{int}\, C \cup \mathrm{int}\, D)) \\
&= \mathrm{cl}((\,\mathrm{int}\, A \cap \mathrm{int}\, C) \cup (\,\mathrm{int}\, A \cap \mathrm{int}\, D)) \\
&= \mathrm{cl}(\,\mathrm{int}\, A \cap \mathrm{int}\, C) \cup \mathrm{cl}(\,\mathrm{int}\, A \cap \mathrm{int}\, D) \\
&= \mathrm{cl}(\,\mathrm{int}(A \cap C)) \cup \mathrm{cl}(\,\mathrm{int}\, A \cap D)) \\
&= (A \wedge C) \cup (A \wedge D) \\
&= (A \wedge C) \vee (A \wedge D).
\end{aligned}$$

<u>Elementary Relations in a Boolean Algebra</u>: In this section we shall discuss some of the elementary relations that hold in Boolean algebras. However, we shall later prove the representation theorem for Boolean

algebras, the use of which reduces all elementary relations to set-theoretic trivialities.

Remark: The following relations hold in every Boolean Algebra

(1) $x \vee y = x \vee (x' \wedge y)$

(2) $x \wedge y = x \wedge (x' \vee y)$

(3) $x \leq y$ if and only if $x \wedge y' = 0$ if and only if $x' \vee y = 1$

Proofs: (1) $x \vee (x' \wedge y) = (x \vee x') \wedge (x \vee y) = 1 \wedge (x \vee y) = x \vee y$

(2) $x \wedge (x' \vee y) = (x \wedge x') \vee (x \wedge y) = 0 \vee (x \wedge y) = x \wedge y$

(3) $x \leq y$ iff $x \wedge y = x$ iff $x \wedge y \wedge y' = x \wedge y'$ iff $x \wedge 0 = x \wedge y'$ iff $0 = x \wedge y'$. The other part of (3) can be done similarly.

Lemma 1: In every Boolean algebra we have $x \leq y$ if and only if $x' \geq y'$.

Remark: In every Boolean algebra, complementation is unique. More explicitly, if x and y are such that $x \wedge y = 0$ and $x \vee y = 1$, then $y = x'$.

Lemma 2: Every Boolean algebra satisfies de Morgan Laws:

(1) $(x \vee y)' = x' \wedge y'$

(2) $(x \wedge y)' = x' \vee y'$.

Ideals and Filters of a Boolean Algebra

Definition: An ideal in a Boolean algebra B is a subset I of B satisfying

(1) $a \vee b$ belongs to I whenever both a and b belong to I.

(2) If b belongs to I and $a \leq b$, then a belongs to I.

It is clear that an ideal is not all of B if and only if the unit element is not in the ideal. Such an ideal is called a proper ideal.

Definition: A filter F in B is a subset of B satisfying

(1) $a \wedge b$ belongs to F whenever both a and b belong to F.

(2) If $b \in F$ and $b \leq a$, then a belongs to F.

A filter is proper, i.e. not all of B, if and only if the zero element is not in the filter.

Since every Boolean algebra satisfies the de Morgan Laws

$$a \vee b = (a' \wedge b')' \qquad \text{and}$$

$$a \wedge b = (a' \vee b')'$$

the complements of members of an ideal form a filter and conversely. For this reason, a filter is frequently called a dual ideal.

A principal ideal I_a is the set of elements less than or equal to a single element a of B. A principal filter is defined similarly. A filter or ideal is called maximal if the only filter or ideal properly containing it is B itself.

Theorem 2: A proper filter (respectively ideal) is maximal if and only if for every element a of B, either a or a' belongs to the filter (respectively ideal).

Proof: If I is a proper ideal $\ni \forall a \in B$, $a \in I$ or $a' \in I$, and J is an ideal with $I \subset J$, then there is an $a \in J$ and $a \notin I$, but $a' \in I$, so $a' \in J$, therefore $a \vee a' = 1 \in J$, hence $J = B$.

Conversely, suppose I is maximal and $a \notin I$. Then the ideal generated by I and $\{a\}$ contains I properly and is therefore all of B. Hence, 1 is the supremum of a and finitely many elements of I. But this implies that a' is dominated by the supremum of finitely many elements of I and therefore belongs to I.

Homomorphisms: A Boolean algebra homomorphism h from a Boolean algebra B to a Boolean algebra C is a function which preserves the Boolean operations i.e.

(1) $h(a \wedge b) = h(a) \wedge h(b)$

(2) $h(a \vee b) = h(a) \vee h(b)$

(3) $h(a') = (h(a))'$

Note that in the presence of (3), and since a Boolean algebra satisfies de Morgan's Laws, conditions (1) and (2) are equivalent. Thus to verify that a function is a Boolean algebra homomorphism, it is sufficient to demonstrate that it satisfies (3) and either (1) or (2). A Boolean algebra homomorphism is called a monomorphism if it is one-to-one. An isomorphism is a monomorphism which is onto.

Theorem 3: A Boolean algebra homomorphism h is a monomorphism if and only if $h(a) = 0$ implies $a = 0$.

Proof: Necessity is clear. To show sufficiency, assume that $h(a) = h(b)$, then $h(a \wedge b') = h(a) \wedge (h(b))' = 0$. Hence $a \wedge b' = 0$. Similarly $h(a \vee b')' = (h(a) \vee (h(b))')' = 1' = 0$. Hence $(a \vee b')' = 0$, so $a \vee b' = 1$. Since complementation in a Boolean algebra is unique, we have $a' = b'$, and $a = b$.

Fields of Sets

Definition: A field of sets is a family of subsets of a set X which is closed under finite unions, finite intersection and complementation.

Example: the family of clopen subsets of any topological space is a field of sets.

Definition: A field of subsets of X is called a reduced field if for every pair of distinct points of X, there is a member of the field containing one of the points but not the other.

Definition: A filter (respectively ideal) in a field is said to be determined by a point if it is the set of all members of the field containing (respectively not containing) the point.

Lemma 3: Every filter or ideal determined by a point is necessarily maximal.

Proof: Let the filter F be determined by the point $x \in X$ and let A be any subset in the field of sets, then either $x \in A$ or $x \in A'$. Hence either $A \in F$ or $A' \in F$. Therefore F is maximal.

Definition: A perfect field is a field in which every maximal filter, or equivalently every maximal ideal, is determined by a point.

Continuous Functions: Let X be any topological space. The family of all continuous real-valued functions on X will be denoted by $C(X)$. The subset of $C(X)$ consisting of all bounded functions will be denoted by $C^*(X)$. We define addition and multiplication in $C(X)$ as follows: $(f + g)(x) = f(x) + g(x)$, and $(fg)(x) = f(x)\,g(x)$ $\forall$ $f, g \in C(X)$. It can be verified that under these operations $C(X)$ will be a commutative ring with unity.

Zero and Cozero Sets

Definition: A subset of a space X of the form

$$f^{-1}(0) = \{x \in X : f(x) = 0\} \qquad (f \in C(X))$$

will be called the zero-set of f and will be denoted by $Z(f)$. Any set that is a zero-set of some function in $C(X)$ is called a zero-set in X. The complement of a zero-set is called a cozero-set. Zero-sets are closed and cozero-sets are open.

Remark: The family of zero-sets is closed under finite unions and intersections.

Proof: $Z(f) \cup Z(g) = Z(fg)$ and

$$Z(f) \cap Z(g) = Z(|f| + |g|).$$

Completely Separated Sets

Definition: Two sets A and B are said to be completely separated (from one another) in X if there exists a function f in $C^*(X)$ such that $0 \leq f \leq 1$ and $f(x) = 0$ for all $x \in A$, and $f(x) = 1$ for all $x \in B$.

It is enough to find a function g in $C(X)$ satisfying $g(x) \leq 0$ for all $x \in A$ and $g(x) \geq 1$ for all $x \in B$, for then $(0 \vee g) \wedge 1$ has the required properties.

When a zero-set Z contains a neighborhood of a set A, we refer to Z as a zero-set neighborhood of A.

Theorem 4: Two sets are completely separated if and only if they are contained in disjoint zero-sets.

Proof: Sufficiency. If $Z(f) \cap Z(g) = \emptyset$, then $|f| + |g|$ has no zeros, and we may define

$$h(x) = \frac{|f(x)|}{|f(x)| + |g(x)|} \qquad (x \in X) .$$

Then $h \in C(X)$, and h is equal to 0 on $Z(f)$ and to 1 on $Z(g)$.

Conversely, if A and A' are completely separated, there exists $f \in C(X)$ equals to 0 on A and to 1 on A'. The disjoint sets $F = \{x: f(x) \leq \frac{1}{3}\}$, $F' = \{x: f(x) \geq \frac{2}{3}\}$ are zero-set neighborhoods of A and A'. respectively.

Lemma 4: If A and A' are completely separated, then there exist zero sets F and Z such that

$$A \subset X \setminus Z \subset F \subset X \setminus A'$$

For with A, A', f, and F as above we simply take

$$Z = \{x: f(x) \geq \frac{1}{3}\}$$

z-filters and z-ultrafilters

Definition: A nonempty subfamily of $Z(X)$ is called a z-filter on X if it is closed under the formation of finite intersections and of supersets, and that does not contain the empty set.

A z-ultrafilter is a maximal z-filter. A prime z-filter is a z-filter F with the property that whenever the union of two zero-sets belongs to F, then at least one of them belongs to F.

Completely Regular Spaces

Definition: A space X is said to be completely regular provided that it is Hausdorff and whenever S is a closed set and x is a point in its complement, there exists a function $f \in C(X)$ such that $f(x) = 1$ and $f(S) = \{0\}$, i.e. S and $\{x\}$ are completely separated.

<u>Theorem 5</u>: A Hausdorff space X is completely regular if and only if the family $Z(X)$ of all zero-sets is a base for the closed sets.

The following results will also prove to be useful:

(1) In a completely regular space every closed set S is an intersection of zero-set neighborhood of S.

(2) In a completely regular space every neighborhood of a point contains a zero-set neighborhood.

From now on all spaces mentioned will be presumed to be completely regular Hausdorff spaces.

Convergence of z-filters

<u>Definition</u>: A point $p \in X$ is said to be a cluster point of a z-filter F if $p \in \bigcap F$.

<u>Definition</u>: A z-filter F is said to converge to the limit p if every neighborhood of p contains a member of F.

Lemma 5: If F converges to p, then p is a cluster point of F.

Lemma 6: A z-filter F converges to p if and only if F contains the z-filter of all zero-set neighborhoods of p.

Corollary: A z-ultrafilter converges to any of its cluster points.

Proof: Let p be a cluster point of a z-ultrafilter F and E be the z-filter of all zero-set neighborhoods of p. Then $F \cup E$ has the finite intersection property and hence is embeddable in a z-ultrafilter. Therefore $E \subset F$.

Lemma 7: Let F be a prime z-filter on X and p a cluster point of F, then F converges to p.

Proof: Let U be any zero-set neighborhood of p. Since X is completely regular, U contains a neighborhood of p of the form $X \setminus Z$, where Z is a zero-set. Since $U \cup Z = X \varepsilon F$, either U or Z belongs to the prime z-filter F. But $Z \notin F$ because $p \notin Z$. So $U \varepsilon F$ and F converges to p.

Corollary: In a compact completely regular space X, every prime z-filter is convergent.

Lemma 8: A z-filter has at most one limit.

The family of all zero-sets containing a given point p is denoted by F_p. Because any zero-set not containing p is completely separated from $\{p\}$, F_p is actually a z-ultrafilter. We call a z-filter free or fixed according as the intersection of all its members is empty or nonempty.

Corollary: There is a one-to-one onto map between the completely regular Hausdorff space X and the family of all fixed z-ultrafilters such that each fixed z-ultrafilter converges to the corresponding point.

Proof: Let $\psi : p \longrightarrow F_p$. It is clear that ψ is well defined. ψ is one-to-one because a z-ultrafilter has at most one limit. ψ is onto for if a z-ultrafilter F converges to the point q, then $F \subset F_q$ and hence $F = F_q$.

Let X be dense in a space H, and consider a z-filter F on X. A point $p \in H$ is a cluster point of F if every neighborhood (in H) of p meets every member of F. Hence p is a cluster point of F if and only if

$$p \in \bigcap_{Z \in F} cl_H Z$$

A z-filter F converges to the limit p if every neighborhood (in H) of p contains a member of F.

Remark: It is easily seen that limits are unique, and that a z-ultrafilter converges to any of its cluster points.

Lemma 9: If X is dense in H, then every point of H is the limit of at least one z-ultrafilter on X.

Proof: Let $p \in H$ and E be the z-filter on H of all zero-set neighborhoods (in H) of p. Let B be the trace of E on X. Since $p \in \text{cl}\, X$, $B \cup \{X\}$ has the finite intersection property and hence is embeddable in a z-ultrafilter F. Clearly, F converges to p.

The Mapping τ^*: Let τ be a continuous mapping from X into Y, and let F be a z-filter on X. Define

$$\tau^*F = \{Z \in Z(Y) : \tau^{-1}[Z] \in F\}.$$

It is easily seen that τ^*F is a z-filter on Y.

Lemma 10: If F is prime, then τ^*F is also prime.

Definition: A topological space is said to be totally disconnected if the only connected subsets are the singletons.

Definition: A space is said to be zero-dimensional if it has a base of clopen sets.

<u>Lemma 11</u>: Every zero-dimensional space is totally disconnected.

<u>Proof</u>: Let X be zero-dimensional and let S be a subset containing at least two points x and y. There is a clopen set U containing x but not y because X is Hausdorff. The set $U \cap S$ is proper and clopen in S. Hence S is disconnected.

The converse of the Lemma is true for compact spaces. A couple of results are needed to establish the converse.

<u>Definition</u>: By a partition of a subset S of a space X we shall mean a finite collection of disjoint, relatively clopen subsets of S whose union is S.

<u>Lemma 12</u>: Let M be a family of compact sets in a compact Hausdorff space X, and let $\{M_1, M_2\}$ be a partition of $\bigcap M$. Then there exists a finite subfamily N of M, and a partition $\{N_1, N_2\}$ of $\bigcap N$ such that $M_1 \subset N_1$ and $M_2 \subset N_2$.

<u>Proof</u>: Since the disjoint sets M_1 and M_2 are closed in the compact Hausdorff space X, they are compact and hence are contained in disjoint open sets U_1 and U_2 respectively. Since $\bigcap M \subset U_1 \cup U_2$ the complements of the members of M form an open cover for the compact

set $X \setminus (U_1 \cup U_2)$ and hence there is a finite subfamily N of M such that $\bigcap N \subset U_1 \cup U_2$. Then the sets

$$N_1 = (\bigcap N) \cap U_1 \quad , \quad N_2 = (\bigcap N) \cap U_2$$

form a partition of $\bigcap N$ with $M_1 \subset N_1$ and $M_2 \subset N_2$.

Definition: The component of a point is the maximal connected set containing it.

Since an arbitrary union of connected sets with a point in common is connected, it follows that the component of a point is the union of all the connected sets containing it.

Theorem 6: The component of a point in a compact space is the intersection of all the clopen sets containing it.

Proof: Let M denote the family of all clopen sets containg a point x. We will show that $C = \bigcap M$ is connected. Let $\{M_1, M_2\}$ be a partition of C, with $x \in M_1$, and let N_1 and N_2 be as in the last Lemma. Since $N_1 \cup N_2$ is the intersection of a finite family of clopen sets, it is itself clopen. But since N_1 and N_2 are clopen in $N_1 \cup N_2$, each is clopen in X. Since $x \in M_1 \subset N_1$, N_1 is a member of M and hence $C \subset N_1$. Therefore $C = C \cap N_1 = M_1$, and so M_2 is empty, thus C is connected.

<u>Theorem 7</u>: A compact, totally disconnected space is zero-dimensional.

<u>Proof</u>: Let X be a compact totally disconnected space and let U be an open set containing the point x. Since X is totally disconnected, the singleton $\{x\}$ is a component in X. Let $\{U_\alpha : \alpha \in I\}$ be the family of all clopen sets containing x. Since X is compact, we have $\{x\} = \bigcap \{U_\alpha : \alpha \in I\}$. If y is any point distinct from x, then there is a clopen set U_α such that $x \in U_\alpha$ but $y \notin U_\alpha$. Hence $\{U\} \cup \{X \setminus U_\alpha : \alpha \in I\}$ is an open cover for X and $\{U\} \cup \{X \setminus U_{\alpha_i},\ i = 1, 2, \cdots, n\}$ is a finite subcover. So $X \setminus U \subseteq \bigcup (X \setminus U_{\alpha_i})$ and hence $\bigcap U_{\alpha_i}$ is a clopen neighborhood of x contained in U. Therefore, X has a base of clopen sets.

CHAPTER 2

THE STONE-ČECH COMPACTIFICATION

Let X be a topological space. By a compactification of X we mean a pair (Y, h) consisting of a compact Hausdorff space Y and a homeomorphism h of X onto a dense subset of Y. We shall also say that Y is a compactification of X, indicating that X is homeomorphic to a dense subspace of Y. Since a subspace of a compact Hausdorff space is necessarily completely regular, only completely regular spaces can be compactified.

In this chapter we consider the Stone-Čech compactification βX of a completely regular Hausdorff space X. The existence and uniqueness of βX were first proved by M.H. Stone utilizing the theory of representation of topological spaces as maps in Boolean spaces. A second, simpler, proof was given by Čech. A third construction of β, valid for normal spaces only, was obtained by Wallman. A. Weil presented a construction based on the theory of uniform structures. In 1941, Gelfand and Shilov presented a simplified version of Stone's original construction. Kakutani gave a construction of β using Banach lattices.

<u>Theorem 1 (Stone-Čech)</u>: Every completely regular Hausdorff space X

has a Hausdorff compactification βX in which completely separated subsets of X have disjoint closures in βX.

Proof: (i) Definition of βX

Since X can be put into one-to-one correspondence with the family of all fixed z-ultrafilters, it serves as an index set for that family. We increase X to an index set βX for the family of all z-ultrafilters.

(ii) Definition of the Topology on βX.

In the remainder of this proof Z will denote a zero-set in the space X. Define

$$\overline{Z} = \{p \in \beta X : Z \in F_p\} ,$$

that is $p \in \overline{Z}$ if and only if $Z \in F_p$. In particular, $\overline{X} = \beta X$ and $\overline{\phi} = \phi$. Now since $Z_1 \cup Z_2 \in F_p$ if and only if $Z_1 \in F_p$ or $Z_2 \in F_p$, then $\overline{Z}_1 \cup \overline{Z}_2 = \overline{Z_1 \cup Z_2}$. Thus, the family of sets $\overline{Z}$ is closed under finite union and contains the empty set and hence may serve as a base for the closed sets in βX.

(iii) X is a subspace of βX.

$p \in \overline{Z} \cap X$ iff $Z \in F_p$ iff $p \in Z$. Hence $\overline{Z} \cap X = Z$.

Hence the induced topology of X coincides with the original topology on X.

(iv) X is dense in $\beta\overline{X}$.

More generally we will show

$$cl_{\beta X} Z = \overline{Z}$$

from which follows $cl_{\beta X} X = \overline{X} = \beta X$. Now $cl\ Z \subset \overline{Z}$ because $Z \subset \overline{Z}$. On the other hand, for every basic closed set $\overline{Z'}$ containing Z, we have $Z' = \overline{Z'} \cap X \supset Z$, hence $\overline{Z'} \supset \overline{Z}$. Therefore $cl\ Z \supset \overline{Z}$.

(v) βX is Hausdorff

Let p and q be any two distinct points in βX. Choose disjoint zero-sets $A \in F_p$ and $A' \in F_q$. There exists a zero-set Z disjoint from A, and a zero-set Z' disjoint from A', such that $Z \cup Z' = X$. Since $Z \notin F_p$ and $Z' \notin F_q$, then $p \notin cl\,Z$ and $q \notin cl\,Z'$. Since $cl\ Z \cup cl\ Z' = \beta X$, the neighborhoods $\beta X \setminus cl\ Z$ of p and $\beta X \setminus cl\ Z'$ of q are disjoint.

(vi) For any two zero-sets Z_1 and Z_2 in X,

$$cl_{\beta X} (Z_1 \cap Z_2) = cl_{\beta X} Z_1 \cap cl_{\beta X} Z_2 .$$

Since $cl_{\beta X} Z = \overline{Z}$, we have $p \in cl_{\beta X} Z$ iff $Z \in F_p$. Also since $Z_1 \cap Z_2 \in F_p$ iff $Z_1 \in F_p$ and $Z_2 \in F_p$, it follows that

$$cl_{\beta X} (Z_1 \cap Z_2) = cl_{\beta X} Z_1 \cap cl_{\beta X} Z_2 .$$

Now since completely separated sets in X are contained in disjoint zero-sets in X, it follows that they have disjoint closures in βX.

(vii) <u>βX is compact</u>

Let $\Phi = \{cl\ Z : Z \in B\}$ be a family having the finite intersection property. By (vi), B itself has the finite intersection property and hence is embeddable in a z-ultrafilter F_p and we have

$$p \in \bigcap_{Z \in F_p} cl\ Z \subset \bigcap_{Z \in B} cl\ Z .$$

So Φ has a nonempty intersection and hence βX is compact.

<u>Proposition 1</u>: Every point of βX is the limit of a unique z-ultrafilter on X.

<u>Proof</u>: Since X is dense in βX, each point of βX is the limit of at least one z-ultrafilter on X. Let F_1 and F_2 be distinct z-ultrafilters converging to a common limit p, then there exist disjoint zero-sets $Z_1 \in F_1$ and $Z_2 \in F_2$ where

$p \in cl_{\beta X} Z_1 \cap cl_{\beta X} Z_2$. This is a contradiction to part (iv) in the proof of the previous theorem.

Theorem 2 (M.H. Stone): Every continuous mapping τ from X into a compact space Y has a unique extension $\bar{\tau}$ from βX into Y.

Proof: We will construct the map $\bar{\tau} : \beta X \longrightarrow Y$. Let $p \in \beta X$ and let F denote the unique z-ultrafilter on X with limit p. Write

$$\tau^*F = \{C \in Z(Y) : \tau^{-1}[C] \in F\} .$$

By Lemma 10 in Chapter 1, τ^*F is a prime z-filter and since Y is compact, τ^*F has a limit in Y. Denote this limit by $\bar{\tau}p$, that is

$$\bigcap \tau^*F = \{\bar{\tau}p\} .$$

We have defined the mapping $\bar{\tau}$ from βX into Y. If $p \in X$, then $p \in \bigcap F$ and hence $\tau p \in \bigcap \tau^*F$. Therefore $\bar{\tau}|X = \tau$.

Before establishing the continuity of τ we need a short remark. Let C and $C' \in Z(Y)$, and let $Z = \tau^{-1}[C]$ and $Z' = \tau^{-1}[C']$. If $p \in cl_{\beta X} Z$, then $Z \in F_p$ and hence $C \in \tau^*F_p$. That is $p \in cl_{\beta X} Z$ implies $\bar{\tau}p \in C$.

To prove that $\bar{\tau}$ is continuous, let C be a zero-set neighborhood of $\bar{\tau}p$. By Lemma 4 in Chapter 1, there is a zero-set C' whose complement is a neighborhood of $\bar{\tau}p$ contained in C. Then $C \cup C' = Y$, so that $Z \cup Z' = X$, and hence $cl_{\beta X} Z \cup cl_{X\beta} Z' = \beta X$. Since $\bar{\tau}p \notin C'$, then $p \notin cl_{\beta X} Z'$. Therefore $\beta X \setminus cl_{\beta X} Z'$ is a neighborhood of p and such that $\bar{\tau}[\beta X \setminus cl_{\beta X} Z']$ C.

The uniqueness of the extension $\bar{\tau}$ follows from the fact that any two extensions of τ must agree on the dense subspace X of βX.

<u>Theorem 3(Stone-Cech)</u>: The space X is C^*-embedded in its compactification βX.

<u>Proof</u>: Any function g in $C^*(X)$ is a continuous mapping into the compact subset $cl_R g[X]$ of R. By Theorem 2, g has an extension from βX into $cl_R g[X]$.

<u>Corollary 1</u>: Any compactification of X to which every mapping of X to a compact space has an extension is homeomorphic to βX under a homeomorphism which leaves points of X fixed.

<u>Proof</u>: Let H be a compactification of X satisfying the given factorization condition and consider the following commutative diagram.

$$\begin{tikzcd}
 & X \arrow[dl, "i"'] \arrow[dr, "i"] & \\
H \arrow[rr, shift left, "f'"] & & \beta X \arrow[ll, shift left, "f"]
\end{tikzcd}$$

The identity mapping i on X is a continuous mapping into the compact space H and hence has an extension $f : \beta X \longrightarrow H$. Similarly the same mapping has an extension $f' : H \longrightarrow \beta X$. Since $(f' \cdot f)|X$ is the identity on X and since X is dense in βX, it follows that $f' \cdot f$ is the identity on βX. Similarly $f \cdot f'$ is the identity on H. Hence f and f' are homeomorphism leaving points of X fixed.

A partial ordering on the set of all compactifications of a space X can be defined in the following manner: If H_1 and H_2 are two compactifications of X, write $H_1 \leq H_2$ whenever there exists a mapping f from H_2 onto H_1 which leaves the points of X fixed.

<u>Proporiton 2</u>: βX is a maximal element in the set of all compactifications of a completely regular Hausdorff space X.

<u>Proof</u>: Let H be any compactification of X and consider the following diagram:

$$\begin{tikzcd} & X \arrow[dl, "i"'] \\ H & \beta X \arrow[l, two heads, "\bar{i}"] \end{tikzcd}$$

The identity map i from X into H has a Stone extension $\bar{i}$ from βX into H. Since βX is compact, $\bar{i}[\beta X]$ is compact and hence closed in H. Since X is dense in H and is contained in $\bar{T}[\beta X]$, it follows that $\bar{T}[\beta X] = H$ and hence $\bar{T}$ is onto.

Let X be dense in H. The question now arises as to when one can identify H with a subspace of βX containing X. It comes out that this identification is possible if and only if X is C*-embedded in H.

Proposition 3: Let X be dense in H. The following are equivalent:

(i) X is C*-embedded in H

(ii) $\beta H = \beta X$

(iii) $X \subset H \subset \beta X$.

Proof: (i) $\Longrightarrow$ (ii) Since X is dense in H and H is dense in βH, then X is dense in βH. Moreover, X is C*-embedded in H and H is C*-embedded in βH, hence X is C*-embedded in βH. By Corollary 1, $\beta H = \beta X$.

(ii) $\Longrightarrow$ (iii). $X \subset H \subset \beta H = \beta X$.

(iii) $\Longrightarrow$ (i). Since X is C*-embedded in βX and $X \subset H \subset \beta X$ it follows that X is C*-embedded in H.

Lemma 1: The direct product of completely regular spaces is completely regular.

Proof: Let $X = \prod_\alpha X_\alpha$, where each X_α is completely regular. Let e_α be the projection map from X onto X_α. Since the family of zero-sets of X_α forms a basis for the closed sets in X_α, the collection of all finite unions

$$e_{\alpha_1}^{-1}[Z_1] \cup \cdots \cup e_{\alpha_n}^{-1}[Z_n] \quad ,$$

where Z_k is a zero-set in X_{α_k}, is a base for the closed sets in X. Since each such union is a zero-set, X is completely regular.

Remark: A mapping f from a space into a product $X = \prod_\alpha X_\alpha$ is continuous if and only if $e_\alpha \cdot f$ is continuous for each projection e_α.

Now let $X = \prod_\alpha X_\alpha$, where each X_α is compact. Then X is completely regular. Each projection $e_\alpha : X \longrightarrow X_\alpha$ has a Stone extension $\overline{e}_\alpha : \beta X \longrightarrow X$. By the previous remark, the mapping $f : \beta X \longrightarrow X$ defined by $f(p) = (\overline{e}_\alpha P)_\alpha$ is continuous. Since f is onto, we conclude that X is compact iff each X_α is compact.

E. Hewitt believed at a time that the product $\prod_\alpha \beta X_\alpha$ of completely regular spaces X_α is homeomorphic to $\beta(\prod_\alpha X_\alpha)$ [13, Theorem 14]. This was later shown to be untrue in general and was corrected by I. Glicksberg as will be shown in the remainder of this chapter.

For any two completely regular spaces X and Y, their product $X \times Y$ is dense in $\beta X \times \beta Y$. Hence $\beta X \times \beta Y$ is a compactification of $X \times Y$. The question now arises as to when one can identify this with the Stone-Čech compactification. It has been shown by I. Glicksberg that a side from a trivial case, this identification is possible if and only if $X \times Y$ is pseudocompact, i.e. if and only if every real valued continuous function on $X \times Y$ is bounded. Some characteristics of pseudocompact spaces will be used and we study them here. A family of open subsets of a space X is called locally finite if every point of X has a neighborhood that meets only finitely many members of the family. A point in the space X is said to be a cluster point of a family of open subsets of X if every neighborhood of the point meets infinitely many members of the family.

Lemma 2: A space X is pseudocompact if and only if every sequence of nonvoid open subsets has a cluster point.

Proof: Let X be a pseudocompact space that admits a sequence $\{U_n\}$ of nonvoid open sets with no cluster point. Then $\{U_n\}$ is locally finite. Since each U_n is infinite we can choose a sequence $\{P_n\}$ of distinct points such that $P_n \in U_n$. Now since for each n, $X \setminus U_n$ is contained in a zero-set we can choose a continuous function f_n such that

$$\begin{cases} f_n(P_n) = n \\ f_n(x) = 0 \quad \text{for all} \quad x \in X \setminus U_n \end{cases}$$

Now let $f = \Sigma f_n$. Since $\{U_n\}$ in locally finite, the value of f at each point x in X is a finite sum of values of f_n's at x, hence f is continuous. But f is unbounded, a contradiction to the pseudocompactness of X.

Conversely, suppose X is not pseudocompact, then there exists an unbounded function f on X such that $f[X]$ contains a sequence $\{m_n\}$ of real numbers that has no accumulation point in R. Now we can choose a locally finite collection of open intervals $\{U_n\}$ in R such that $m_n \in U_n$. Thus the family $\{f^{-1}(U_n)\}$ is a locally finite sequence of nonvoid open sets in X. Hence there is a sequence of nonvoid open sets with no cluster point.

Lemma 3: A space X is pseudocompact if and only if every bounded function on X assumes its least upper bound.

Proof: Suppose X is pseudocompact and let $f \in C(X)$, then $f[X]$ is pseudocompact because a continuous image of a pseudocompact space is pseudocompact. If $f[X]$ does not assume its sup m then we can define an unbounded function g on $f[X]$ by

$$g(t) = \frac{1}{m - t} \quad \text{where} \quad t \in f[X] .$$

This is a contradiction to the pseudocompactness of $f[X]$.

Conversely, suppose that each bounded function on X assumes its least upper bound and let f be an unbounded function on X. Define the function $g \in C(X)$ by

$$g = \frac{-1}{1 + |f|}$$

Then $-1 \leq g < 0$ and g has sup 0. Obviously, g does not assume its sup which is a contradiction to the assumption. Hence X is pseudocompact.

<u>Lemma 4</u>: A space X is pseudocompact if and only if there is no nonvoid closed G_δ in $\beta X \setminus X$.

<u>Proof</u>: From the last Lemma, if X is pseudocompact then $g[X]$ assumes its least upper bound for any $g \in C(X)$.

Suppose there exists a closed nonvoid $G_\delta = \bigcap U_n$, where U_n is open in βX. Let $C_n = \beta X \setminus U_n$. Since G_δ is compact and C_n is closed they are completely separated. Hence there exists $f_n \in C(\beta X)$, for each n with $0 \leq f_n \leq 1$ such that $f_n | G_\delta = 1$ and $f_n | C_n = 0$. Define $g_n = f_n \wedge 2^{-n}$. Then $0 \leq g_n \leq 2^{-n}$. Let $g(x) = \sum_{n \in N} g_n(x)$, for each $x \in \beta X$. Then $g \in C(\beta X)$. It is clear that $\sup g[\beta X] = 1 = \sup g[X]$. Now let $x' \in X$, then there exists a number m such that $x' \notin U_m$, because $G_\delta \cap X = \phi$. Since $g_m(x') = 0$ we have $g(x') = \sum_{n=1}^{\infty} g_n(x') \leq 1 - 2^{-m}$. Note that $g|X \in C(X)$. Since X is pseudocompact, $g[X]$ assumes sup $g[X]$. This is a contradiction because g assumes sup for no $x' \in X$.

Conversely, suppose X is not pseudocompact, then by the Lemma 3 there is a bounded function f on X that does not assume its least upper bound m. Since X is dense in βX the number m is also the least upper bound of the Stone extension $\overline{f}$ to βX. Hence the set

$$\overline{f}^{-1}(m) = \bigcap_{k=1}^{\infty} \{\overline{f}^{-1}(m - \frac{1}{k}, m + \frac{1}{k})\}$$

is a nonvoid closed G_δ in $\beta X \setminus X$.

<u>Lemma 5</u>: If X is pseudocompact, then a bounded set of functions on X is equicontinuous if and only if each countable subset is equicontinuous.

The proof of this result is well known in [9,10]

<u>Lemma 6</u>: If $X \times Y$ is pseudocompact and $f \in C(X \times Y)$, then the family $\{f(x,\cdot) : x \in X\}$ is equicontinuous on Y. Hence the mapping $y \to f(\cdot,y)$ of Y into $C(X)$ is continuous.

<u>Proof</u>: Since Y is pseudocompact it is sufficient to show that any sequence $\{f(x_n',\cdot)\}$ is equicontinuous on Y. Suppose not, then for

some y_0 in Y and $\varepsilon > 0$, no neighborhood U of y_0 satisfies the condition.

$$(*) \qquad |f(x_n',y) - f(x_n',y_0)| < \varepsilon \quad \text{for} \quad y \in U,$$

for all n. We can now select a subsequence $\{x_n\}$ of $\{x_n'\}$ and open neighborhoods V_n of x_n, U_n of y_0 in the following manner. Let $U_1 = Y$ and x_1 be the first x_n' for which (*) fails for $U = U_1$. We choose a neighborhood $V_1 \times U_2$ of (x_1,y_0) on which f varies by less than 1. Having chosen $x_1,\cdots,x_k$, $V_1,\cdots,V_k$, $U_1,\cdots,U_{k+1}$ we select x_{k+1} as the first x_n' for which (*) fails for $U = U_{k+1}$, and select an open neighborhood $V_{k+1} \times U_{k+2}$ of (x_{k+1}, y_0) on which f varies by $< 1/(k+1)$, with $\mathrm{cl}\, U_{k+2} \subset U_{k+1}$.

From our choice of x_n we have a y_n in U_n for which $|f(x_n, y_n) - f(x_n, y_0)| \geq \varepsilon$; consequently we may choose an open neighborhood $\hat{V}_n \times \hat{U}_n$ of (x_n, y_n) lying within $V_n \times U_n$ which yields $|f(x, y) - f(x, y_0)| > \frac{\varepsilon}{2}$ for $(x,y) \in \hat{V}_n \times \hat{U}_n$. But by Lemma 2, $\{\hat{V}_n \times \hat{U}_n\}$ has a cluster point $(\overline{x}, \overline{y})$ in $X \times Y$, so that $|f(\overline{x}, \overline{y}) - f(\overline{x}, y_0) \geq \frac{\varepsilon}{2}$ by continuity. On the other hand, as a cluster point of $\{\hat{U}_n\}$, $\overline{y} \in \bigcap_{j=1}^{\infty} U_j$; for $\overline{y} \notin U_k$ implies $\overline{y} \notin \mathrm{cl}\, U_{k+1}$ so that $Y \setminus \mathrm{cl}\, U_{k+1}$ is a neighborhood of $\overline{y}$ meeting only finitely many $\hat{U}_n$. Thus for $x \in \hat{V}_n \subset V_n$, $|f(x,\overline{y})-f(x,y_0)| < \frac{1}{n}$

since both $(x,\overline{y})$ and (x,y_0) lie in $V_n \times U_{n+1}$. Since $\overline{x}$ is a cluster point of $\{\hat{V}_n\}$, $0 = |f(\overline{x},\overline{y}) - f(\overline{x},y_0) \geq \frac{\varepsilon}{2}$, a contradiction. Hence $\{f(x,\cdot) : x \in X\}$ is equicontinuous on Y.

To show the second statement, given a positive number ε and a point $y_0 \in Y$, there is an open neighborhood U of y_0 such that

$$|g(x, y_1) - g(x, y_0)| < \frac{\varepsilon}{2}$$

for all $y_1 \in U$ and for each $g \in C(Y)$. Now, let $y_0 \rightarrow f(\cdot, y_0)$ and $y_1 \rightarrow f(\cdot, y_1)$ where $f(\cdot, y_0)$ and $f(\cdot, y_1)$ are in $C(X)$. Then at each $x \in X$ we have

$$|f(x,y_1) - f(x,y_0)| = |g(x,y_1) - g(x,y_0)| < \frac{\varepsilon}{2}$$

for some $g \in C(Y)$. Hence

$$\max\{\,|f(\cdot,y_1) - f(\cdot,y_0)|\,\} < \varepsilon$$

and hence the mapping $y \longrightarrow f(\cdot,y)$ of Y into $C(X)$ is continuous.

<u>Lemma 7</u>: Let X and Y be completely regular and $f \in C(X \times Y)$. If the mapping $y \rightarrow f(\cdot,y)$ of Y into $C(X)$ is continuous,

then f has a continuous extension to $\beta X \times Y$.

Proof: Since $C(X)$ and $C(\beta X)$ are isomorphic, continuity of the mapping $y \to f(\cdot,y)$ implies continuity of the mapping $y \to \overline{f}(\cdot,y)$ of Y into $C(\beta X)$, where $\overline{f}(\cdot,y)$ is the Stone extension of $f(\cdot,y)$ to βX. But $\overline{f}$ is continuous as a function on $\beta X \times Y$. For given $(x_0,y_0) \in \beta X \times Y$ and $\varepsilon > 0$, we have a neighborhood V of x_0 satisfying $|\overline{f}(x,y_0) - \overline{f}(x_0,y_0)| < \varepsilon$ for $x \in V$, since $\overline{f}(\cdot,y_0) \in C(\beta X)$; since $y \to \overline{f}(\cdot,y)$ is continuous, y_0 has a neighborhood U satisfying $|\overline{f}(x,y)-\overline{f}(x,y_0)| < \varepsilon$ for $y \in U$ and all x in βX. Thus for $(x,y) \in V \times U$, we have

$$|\overline{f}(x,y) - \overline{f}(x_0,y_0)| \leq |\overline{f}(x,y) - \overline{f}(x,y_0)| + |\overline{f}(x,y_0) - \overline{f}(x_0,y_0)| < 2\varepsilon,$$

which establishes the continuity of $\overline{f}$ on $\beta X \times Y$.

Remark: If $X \times Y$ forms a finite set, then every $f \in C(X \times Y)$ clearly has a continuous extension to $\beta X \times \beta Y$, and we have $\beta(X \times Y) = \beta X \times \beta Y$.

Theorem 4: If X and Y are infinite completely regular spaces, then $\beta(X \times Y) = \beta X \times \beta Y$ if and only if $X \times Y$ is pseudocompact.

Proof: (Sufficiency). If $X \times Y$ is pseudocompact, then by our last two Lemmas we may extend $f \varepsilon C(X \times Y)$ continuously to $\beta X \times Y$. Now any space containing a dense pseudocompact subspace is pseudocompact (since any continuous function on the space, being bounded on the dense subspace, is bounded); thus $\beta X \times Y$ is pseudocompact and applying the Lemmas once more we obtain a continuous extension to $\beta X \times \beta Y$. Therefore $\beta(X \times Y) = \beta X \times \beta Y$.

(Necessity). Suppose that $\beta(X \times Y) = \beta X \times \beta Y$, we need to show that $X \times Y$ is pseudocompact. We shall first show that each of the spaces X and Y is pseudocompact. In showing that Y is pseudocompact, we may assume X is compact. For if $f \varepsilon C(\beta X \times Y)$, then the Stone extension of $f|(X \times Y)$ clearly must coincide with f on $\beta X \times Y$, and thus continuously extends f to $\beta X \times \beta Y$; thus $\beta(\beta X \times Y) = \beta X \times \beta Y$, and we may replace X by βX.

Suppose now that X is compact and that Y is not pseudocompact. Then there is a function $g \varepsilon C(Y)$ which never vanishes on Y but has zero as its greatest lower bound, and hence has a sequence of values $g(y_n) \longrightarrow 0$. Since X is infinite, there is a function $f \varepsilon C(X)$ having infinitely many values, and we may assume that a sequence of these, $\{f(x_n)\}$, strictly decreases to zero. By linear interpolation, we can construct a non-negative, continuous, bounded function h on the reals for which $h(f(x_n)) = g(y_n)$, and setting

$$\alpha(x,y) = 2h(f(x))\, g(y) / (h(f(x))^2 + g(y)^2),$$

we obtain an element α of $C(X \times Y)$: for g never vanishes and $0 \leqslant \alpha \leqslant 1$. But consider the Stone extension $\bar{\alpha}$. If (x,y) is a cluster point of the sequence $\{(x_n, y_n)\}$ in the compact space $X \times \beta Y$, then $\bar{\alpha}(x,y) = 1$ since $\bar{\alpha}(x_n, y_n) = \alpha(x_n, y_n) = 1$. On the other hand, x is a cluster point of $\{x_n\}$ and, since $\bar{\alpha}(x_n, y') = \alpha(x_n, y') \rightarrow 0$ for each y' in Y, we have $\bar{\alpha}(x,y') = 0$ for each y' in Y. Since Y is dense in βY, it follows that $\bar{\alpha}(x,y) = 0$, a contradiction. Therefore, Y is pseudocompact.

Now, in order to show that $X \times Y$ is pseudocompact, we need only prove that any closed G_δ in $\beta(X \times Y) \setminus (X \times Y)$ must be void. But if S is such a subset, then for each x in X, $S \cap (\{x\} \times \beta Y)$ is a closed G_δ in $\beta(X \times Y) \setminus \beta(\{x\} \times Y) = \beta(X \times Y) \setminus (\{x\} \times \beta Y)$, and thus is void since $\{x\} \times Y$ is pseudocompact. Hence $S \cap (X \times \beta Y)$ is void and $S \subset (\beta X \setminus X) \times \beta Y$. But now for every y in βY, $S \cap (\beta X \times \{y\})$ is a closed G_δ in $\beta(X \times Y) \setminus \beta(X \times \{y\}) = \beta(X \times Y) \setminus (\beta X \times \{y\})$, hence void, so that $S = S \cap (\beta X \times \beta Y)$ must be void. Therefore $X \times Y$ is pseudocompact.

CHAPTER 3

BOOLEAN ALGEBRA, QUOTIENT EXTENSION RINGS

Every field of sets is a Boolean algebra. In this chapter we will develop the Stone representation theorem which asserts that every Boolean algebra is isomorphic to some field of subsets of some set, namely the clopen subsets of a compact totally disconnected space.

A field of subsets of a set X is called reduced if for every pair of distinct points of X, there is a member of the field containing one of the points but not the other. A perfect field is a field in which every maximal filter is determined by a point, i.e. every maximal filter consists of all members of the field containing a point.

<u>Lemma 1</u>: If E is a perfect reduced field of subsets of a set X, then a topology can be defined on X such that the space X is compact and totally disconnected and E is the Boolean algebra of clopen subsets of X.

<u>Proof</u>: Take E to be the basis for a topology on X. Since every set of E is open in this topology and E is a field, the sets of E are also closed. Since E is a reduced field, for any two points of X, there is a clopen set containing one and missing the other so that X is totally disconnected.

To show that X is compact, we show that any family of closed sets with the finite intersection property has a non-empty intersection. Without loss of generality, let S be such a family of sets of E. Since S has the finite intersection property, it is embeddable in a maximal filter F of E. Now since E is perfect, F is determined by a point $p \in X$. Hence

$$p \in \bigcap F \subset \bigcap S.$$

It remains to prove that every clopen subset V of X belongs to E. The clopen set V is the union of a family $\{E_\alpha\}$ of members of E. But then $\{E_\alpha\} \cup \{X \setminus V\}$ is an open cover of X and has a finite subcover. Hence, V is the union of finitely many sets belonging to E so that V belongs to E.

<u>Lemma 2</u>: In a Boolean algebra every element is contained in a maximal filter.

<u>Proof</u>: Let a be an element of a Boolean algebra B, then the principal filter generated by a is embeddable in a maximal filter.

<u>Proposition 1</u>: Every Boolean algebra is isomorphic to a perfect reduced field of sets.

<u>Proof</u>: Let B be a Boolean algebra and let $S(B)$ denote the set

of all maximal filters of B. Let F be a maximal filter of B and define a map $\phi : B \longrightarrow S(B)$ by

$$\phi(b) = \{ F \in S(B) : b \in F \} .$$

Since $(a \wedge b) \in F$ if and only if $a \in F$ and $b \in F$, $\phi(a \wedge b) = \phi(a) \wedge \phi(b)$. Since for any maximal filter F either $b \in F$ or $b' \in F$ (but not both), $\phi(b') = S(B) \setminus \phi(b)$.

By the last Lemma, every element in B is contained in a maximal filter, hence $\phi(b) = \emptyset$ implies $b = 0$. Thus ϕ is an isomorphism and $E = \{\phi(b) : b \in B\}$ is a field of subsets of $S(B)$ since it is the image of B under ϕ.

If F_1 and F_2 are distinct maximal filters of B, then there is an element $b \in B$ such that $b \in F_1$ but $b \notin F_2$. Hence $F_1 \in \phi(b)$ but $F_2 \notin \phi(b)$. Thus E is a reduced field.

To show that E is a perfect field, let M be a maximal filter of E. Let F be that subset of B such that $F = \phi^{-1}(M)$. F must be a maximal filter of B because ϕ is an isomorphism. If A is any member of M, then $\phi^{-1}(A) \in F$. Hence $F \in \phi(\phi^{-1}(A)) = A$. Hence the maximal filter M is determined by the point F of $S(B)$.

<u>The Stone Representation Theorem</u>: Every Boolean algebra is isomorphic to the Boolean algebra of clopen sets of a compact totally disconnected space.

Proof: Let B be a Boolean algebra, and let $S(B)$ and E be as defined above. By the last proposition B is isomorphic to the reduced perfect field of sets E. By Lemma 1, a topology can be defined on $S(B)$ such that the space $S(B)$ is compact and totally disconnected and E is the Boolean algebra of clopen subsets of $S(B)$.

If D is an infinite discrete space, then it is completely regular and Hausdorff and hence has a Stone-Čech compactification βD. Moreover, the Boolean algebra B of all subsets of D coincides with the family of all zero-sets of D. Hence $S(B)$ is in one-to-one correspondence with the index set βD of all z-ultrafilters on D. Since the same topology has been used to construct the compact spaces $S(B)$ and βD, these two spaces are homeomorphic.

Definition: A space is called extremally disconnected if the closure of every open subset is open.

Proposition 2: The Boolean algebra of clopen subsets of a zero-dimensional space is complete if and only if the space is extremally disconnected.

Proof: Let X be an extremally disconnected zero-dimensional space and let $\{U_\alpha\}$ be a family of clopen subsets of X. The set $C = \mathrm{cl}(\bigcup U_\alpha)$ is open since X is extremally disconnected and hence is an upper bound for $\{U_\alpha\}$. Let C' be any upper bound for $\{U_\alpha\}$, then $\bigcup U_\alpha \subset C'$ and so $C = \mathrm{cl}(\bigcup U_\alpha) \subset C'$. Hence C is the least upper bound for $\{U_\alpha\}$.

Conversely, assume that the Boolean algebra of clopen sets is complete and let V be an open subset of X. Let $\{U_\alpha\}$ be the family of clopen subsets of X contained in V and let U be the least upper bound of $\{U_\alpha\}$. The proof will be complete by showing that $\mathrm{Cl}\ V = U$. Since X is zero-dimensional $V = \bigcup U_\alpha \subset U$. Hence

$$\mathrm{cl}\ V \subset \mathrm{cl}\ U = U.$$

To show the reverse containment, assume that $U \setminus \mathrm{cl}\ V$ is non-empty. Since X is zero-dimensional and the set $U \setminus \mathrm{cl}\ V$ is open, it must contain a non-empty clopen set S. Hence the set $U \setminus S$ is clopen and contains V and so is an upper bound for $\{U_\alpha\}$. This is a contradiction to the assumption that U is the least upper bound. Therefore $\mathrm{cl}\ V = U$.

By a ring we will always mean a ring with unity. If A is a commutative ring, an ideal D in A is called dense if for all $a \in A$, $aD = 0$ implies $a = 0$.

<u>Definition</u>: A ring B $(\supset A)$ is a ring of quotients or rational extension of A provided that for every b, $0 \neq b' \in B$ there exists $a \in A$ such that $ab \in A$ and $ab' \neq 0$. i.e. if $b \neq 0$

in B then $b^{-1}A = \{a \in A : ab \in A\}$ is dense in A. A ring without a proper rational extension is said to be rationally complete.

<u>Definition</u>: Let $\mathbb{D} = \mathbb{D}(A)$ be a family of dense ideals in A. We shall say that the family $\mathbb{D}$ is closed if $A \in \mathbb{D}$ and $DE \in \mathbb{D}$ whenever $D, E \in \mathbb{D}$. We denote, by $\mathbb{D}_0(A)$ the family of all dense ideals in A. Clearly $\mathbb{D}_0$ is closed.

Let $\mathbb{D}$ be a closed family of dense ideals of A. Then the family $\{\mathrm{Hom}_A(D,A)\}_{D \in \mathbb{D}}$ has the following properties

(i) $1 \in \mathrm{Hom}_A(A,A)$

(ii) If $D' \subset D$ in $\mathbb{D}$, $\mathrm{Hom}_A(D,A) \subset \mathrm{Hom}_A(D',A)$

(iii) If f_1, $f_2 \in \mathrm{Hom}_A(D,A)$, then $f_1 \cdot f_2 \in \mathrm{Hom}_A(DD,A)$

Here $f_1 \cdot f_2 = f_1 \circ f_2$.

In what follows $\mathrm{Hom}(D,A)$ will mean $\mathrm{Hom}_A(D,A)$. One can easily see that the system $\{\mathrm{Hom}(D,A)\}_{D \in \mathbb{D}}$ is a direct system if $\mathbb{D}$ is closed. Then the direct limit

$$\varinjlim_{D \in \mathbb{D}} \mathrm{Hom}(D, A) = \bigcup_{D \in \mathbb{D}} \mathrm{Hom}(D, A)/\sim$$

where $\sim$ is the equivalence relation defined for $f_1 \in \mathrm{Hom}(D_1, A)$ and $f_2 \in \mathrm{Hom}(D_2, A)$, $f_1 \sim f_2$ if and only if $f_1 | D_1 D_2 = f_2 | D_1 D_2$.

Let $Q_{\mathbb{D}}(A) = \varinjlim_{D \in \mathbb{D}} \mathrm{Hom}(D,A)$. It is easy to see that $Q_{\mathbb{D}}(A) \subset Q_{\mathbb{D}'}(A)$ for any two closed families $\mathbb{D}$ and $\mathbb{D}'$ with $\mathbb{D} \subset \mathbb{D}'$.

<u>Proposition 3</u>: If $\mathbb{D}$ is a closed family of dense ideals of A, then $Q_{\mathbb{D}}(A)$ is a ring of quotients of A.

<u>Proof</u>: Let f_1 and $0 \neq f_2 \in Q_{\mathbb{D}}(A)$. Let $f_1, f_2 \in \mathrm{Hom}(D,A)$ for some $D \in \mathbb{D}$. Since $f_2 \neq 0$, there exists $d \in D$ such that $f_2(d) \neq 0$. Also note that $d \in \mathrm{Hom}(A,A)$. Since $(f_1 o d)(a) = f_1(da)$ $= f_1(d)a$ for all $a \in A$, thus $f_1 \cdot d = f_1 o d = f_1(d) \in A$.

<u>Proposition 4</u>: If B is a ring of quotients of A, then

$$B = \varinjlim_{D \in \mathbb{D}_0} (B \cap \mathrm{Hom}(D,A))$$

<u>Proof</u>: If $b \in B$, $D \in \mathbb{D}_0$ and $b|D \in \mathrm{Hom}(D,A)$, then $b = b|D \in B \cap \mathrm{Hom}(D,A)$. Thus $B \cap \mathrm{Hom}(D,A)$ has meaning. Finally, each $b^{-1}A$ is dense; thus $b \in \mathrm{Hom}(b^{-1}A, A)$ which shows that $B \subset \bigcup_{D \in \mathbb{D}_0} (B \cap \mathrm{Hom}(D,A))$ and the reverse inclusion is trivial.

<u>Theorem</u>: $Q_{\mathbb{D}_0}(A)$ is the maximal ring of quotients of A.

A proper ideal in a ring is called maximal, if it is not contained in any other proper ideal. A proper ideal P is called prime if for any two elements a and b in A, $ab \in P$ implies $a \in P$ or $b \in P$. The intersection of all maximal ideals of a commutative ring A with unity is called the radical of A, the intersection of all its prime ideals is called the prime radical of A. An element a in a ring is called nilpotent if $a^n = 0$ for some natural number n.

<u>Proposition 5</u>: In a commutative ring A, the prime radical consists of all nilpotent elements of A.

For an ideal K of a commutative ring A we define its annihilator $K^* = \{a \in A : aK = 0\}$. Clearly K^* is an ideal. We also note that $(K_1 + K_2)^* = K_1^* \cap K_2^*$ for any two ideals K_1 and K_2 of A.

<u>Lemma 3</u>: For any ideals J and K of a commutative ring we have

(1) if $J \subset K$ then $K^* \subset J^*$

(2) $K \subset K^{**}$

(3) $K^{***} = K^*$

<u>Corollary</u>: If K is an annihilator ideal then $K^{**} = K$.

<u>Proof</u>: Let $K = J^*$ for some ideal J, then by (3) in the last Lemma $J^{***} = J^*$ and so $K^{**} = K$.

Definition: A commutative ring A with unity is called semisimple if its radical is $\{0\}$ and is called semiprime if its prime radical is $\{0\}$.

Lemma 4: If K is an ideal in a commutative semiprime ring then

(1) $$K \cap K^* = 0$$

(2) $K + K^*$ is dense.

Proof: (1) If $a \in K \cap K^*$ then $a^2 = 0$ and so a is in the prime radical. Hence $a = 0$. (2) If $a(K + K^*) = 0$, then $a \in (K + K^*)^* = K^* \cap K^{**} = 0$ by (1).

In what follows, $J(A) = \{J\colon J \text{ is an ideal in } A \text{ and } J^{**} = J\}$.

Proposition 6: The annihilator ideals in a commutative semiprime ring A form a complete Boolean algebra $J(A)$ with the following properties:

(1) $$J \wedge K = J \cap K$$

(2) $$J' = J^*$$

(3) $$J \vee K = (J^* \cap K^*)^*$$

Proof: The annihilator ideals $J(A)$ are partially ordered by set theoretic inclusion. We proceed to prove that this set is a complemented distributive lattice, i.e. a Boolean algebra.

Since $\bigcap_{\alpha \varepsilon I} J^*_\alpha = (\sum_{\alpha\varepsilon I} J_\alpha)^*$ the intersection of annihilator ideals is an annihilator ideal. Obviously, $(J^* \cap K^*)^*$ is an upper bound for the annihilator ideals J and K. To show that it is the least upper bound we first prove the following

$$J \cap K^* = 0 \iff J \subset K$$

If $J \cap K^* = 0$, then $JK^* = 0$, hence $J \subset K^{**} = K$. Conversely, if $J \subset K$, then $J \cap K^* \subset K \cap K^* = 0$. Now

$$\begin{aligned} J \subset L \text{ and } K \subset L &\iff L^* \cap J = 0 \text{ and } L^* \cap K = 0 \\ &\iff L^* \subset J^* \text{ and } L^* \subset K^* \\ &\iff L^* \subset J^* \cap K^* \\ &\iff (J^* \cap K^*)^* \subset L^{**} = L. \end{aligned}$$

Hence $J \vee K = (J^* \cap K^*)^*$. Also J^* is the complement of J since $J \cap J^* = 0$ and $J \vee J^* = (J^* \cap J^{**})^* = (J^* \cap J)^* = 0^* = 1$. Finally, we show that the distribution law holds. For any annihilator ideals J, K, L and M we have

$$\begin{aligned} J \wedge (K \vee L) \subset M &\iff J \wedge (K \vee L) \wedge M^* = 0 \\ &\iff J \wedge M^* \subset K^* \wedge L^* \end{aligned}$$

$$\iff J \wedge M^* \subset K^* \quad \text{and} \quad J \wedge M^* \subset L^*$$
$$\iff J \wedge M^* \wedge K = 0 \quad \text{and} \quad J \wedge M^* \wedge L = 0$$
$$\iff J \wedge K \quad M \quad \text{and} \quad J \wedge L \subset M$$
$$\iff (J \wedge K) \vee (J \wedge L) \subset M \, .$$

In what follows $Q(A)$ will mean $Q_{D_0}(A)$.

<u>Lemma 5</u>: If D is a dense ideal in A then it is dense in $Q(A)$.

<u>Proof</u>: Let $a \in Q(A)$ and $aD = 0$. If $a \in \mathrm{Hom}(D',A)$ then $a \in \mathrm{Hom}(DD',A)$. If $d \in DD'$, then $(a \circ d)(b) = a(db) = a(d)b$ for all $b \in A$. Hence $a(d) = 0$ and so $a \sim 0$. Therefore $a = 0$.

<u>Lemma 6</u>: If J is an annihilator ideal of $Q(A)$, then $J \cap A$ is an annihilator ideal of A.

<u>Proof</u>: Let J be the annihilator ideal of the ideal K. Then $K \cap A$ is an ideal of A and $J \cap A \subset (K \cap A)^*$. To show the other containment we need to show that for any $a \in A$, if $a \in (K \cap A)^*$ then $a \in K^*$. If $a(K \cap A) = 0$ and $k \in K$, $I = k^{-1}A$, then $kI \in K \cap A$, hence $akI = 0$. Since I is dense $ak = 0$. Thus $aK = 0$ and $a \in K^*$.

<u>Proposition 7</u>: The mapping $\phi : J \longrightarrow J \cap A$ is an isomorphism

of $J(Q(A))$ onto $J(A)$.

Proof: Let J be the annihilator ideal of the ideal $K \subset Q(A)$. By the last Lemma $J \cap A = (K \cap A)^*$. Hence $\phi(J) \varepsilon J(A)$. Moreover, $\phi(0) = 0$, $\phi(J_1 \cap J_2) = (J_1 \cap A) \cap (J_2 \cap A) = \phi(J_1) \cap \phi(J_2)$, and $\phi(J^*) = (J \cap A)^* = \phi(J)^*$. Hence ϕ is a Boolean homomorphism.

ϕ is one-to-one, for if $J \cap A = 0$ then by the last Lemma $J = 0$. It is onto, for if $K \subset A$, then $K^* = J \cap A$ where $J = \{a \varepsilon Q(A) : aK = 0\}$ is an annihilator ideal in $Q(A)$.

Definition: An element $e \varepsilon A$ is called idempotent if $e^2 = e$.

Proposition 8: The idempotents of a ring A with unity form a Boolean algebra A^o.

Proof: For two elements $e, c \varepsilon A^o$ we write $e \leq c$ if $ec = e$. Clearly $(A^o, \leq)$ is a partially ordered set. Also, if $e, c \varepsilon A^o$, then $e \wedge c = ec$ and $e \vee c = e + c - ec$ are the inf and sup respectively. The elements 0 and 1 are in A^o. If $e \varepsilon A^o$, then $e' = 1 - e$ is the complement. It remains to show that A^o is a distributive lattice. We have

$$e \wedge (c \vee d) = e(c + d - cd)$$

$$= ec + ed - ecd$$

$$= ec + ed - eced$$

$$= (e \wedge c) \vee (e \wedge d) .$$

<u>Proposition 9</u>: If A is commutative semiprime and rationally complete, every annihilator K of A can be written as $K = eA$ for some $e \in A^{o}$.

<u>Proof</u>: Since K is an annihilator ideal and A is semiprime, $K + K^*$ is dense in A. Consider $f \in \mathrm{Hom}(K + K^*, A)$ defined by $f(a + b) = a$, where $a \in K$ and $b \in K^*$. Since A is rationally complete there is an element $e \in A$ such that $f(a + b) = e(a + b)$ $= a$. Now $e^2(a + b) = ea = fa = a = e(a + b)$, hence $e^2 - e$ annihilates the dense ideal $K + K^*$ and so $e^2 = e$. Moreover, $K = eK \subset eA$, and similarly $K^* = (1 - e)K^* \subset (1 - e)A$, hence $eA \subset ((1 - e)A)^* \subset K^{**} = K$. Therefore $K = eA$.

<u>Proposition 10</u>: If A is commutative semiprime and rationally complete then $J(A)$ is isomorphic to A^{o}.

<u>Proof</u>: Define $\phi : J(A) \longrightarrow A^{o}$ by $\phi(eA) = e$. ϕ is well defined for if $eA = cA$, then $(e - c)A = 0$ and hence $e = c$. Moreover $\phi((eA)^*) = \phi((1 - e)A) = (1 - e) = e'$. Since $eA \cap cA = ecA$ we have $\phi(eA \cap cA) = \phi(ecA) = ec = e \wedge c$. Hence

ϕ preserves the Boolean operations. ϕ is one-to-one for if $\phi(eA) = \phi(cA)$, then $e = c$ and so $eA = cA$. It is onto for if $e \in A^o$ then $eA = ((1-e)A)^*$ is an annihilator ideal in A.

For a commutative ring A with unity let $M(A)$ denote any set of prime ideals of A. We shall endow $M(A)$ with a topology, which is attributed by some authors to Stone and by others to Zariski.

<u>Proposition 11</u>: $M(A)$ becomes topological space, if as open sets we take all sets of the form

$$\Gamma S = \{P \in M(A) : S \not\subset P\} ,$$

where S is any subset of A. If $M(A)$ contains all maximal ideals, then it is compact.

Note that Γ is a mapping from the set of subsets of A into the set of subsets of $M(A)$. We now define a mapping Δ which goes the other way, by putting

$$\Delta U = \bigcap_{p \in U} P ,$$

for any subset U of $M(A)$.

By the exterior of a subset U we will mean the interior of its complement. A subset of $M(A)$ is called regular open if it is the interior of its closure. Clearly this is the same as saying that it is the interior of some closed set.

Proposition 12: For any subset U of $M(A)$, $\Gamma\Delta U$ is the exterior of U. If $\Delta M(A) = 0$ then, for any subset S of A, $\Delta\Gamma S$ is the annihilator S^* of S.

Proof: To show that $\Gamma\Delta U$ is the exterior of U we need to find for every $P' \in \Gamma\Delta U$ an open set containing P' and not meeting U. We have

$$P' \in \Gamma\Delta U \iff \Delta U \not\subset P'$$

$$\iff \exists_{a \in A} \forall_{p \in U} (a \in P \quad \text{and} \quad a \notin P')$$

$$\iff \exists_{a \in A} (P' \in \Gamma a \quad \text{and} \quad \forall_{p \in U} P \not\subset \Gamma a).$$

Γa is the required neighborhood of P'.

Second, let $a \in \Delta\Gamma M$, we will show that $aM = 0$. We have

$$a \in \Delta\Gamma S \iff \forall_{p \in M(A)} (S \not\subset P \Longrightarrow a \in P)$$

$$\iff \forall_{p \in M(A)} aS \subset P$$

$$\iff aS \subset \Delta M(A)$$

$$\iff aS = 0 .$$

<u>Proposition 13</u>: If $M(A)$ is a prime ideal space of the commutative ring A such that $\Delta M(A) = 0$, then the Boolean algebra of annihilator ideals $J(A)$ of A is isomorphic to the Boolean algebra $R_0(M(A))$ of regular open sets of $M(A)$.

<u>Proof</u>: If J^* is the annihilator ideal of J, then by the last proposition, $J^* = \Delta\Gamma J$ and so $\Gamma J^* = \Gamma\Delta\Gamma A$. Invoking the last proposition again, this means that ΓJ^* is the exterior of ΓA and hence is regular open. Hence the mapping Γ takes an annihilator ideal to a regular open set. It remains to show that Γ is a Boolean isomorphism of $J(A)$ onto $R_0(M(A))$.

Let K be an annihilator ideal in $J(A)$, then $K^* = \Delta\Gamma K$ and so $\Gamma K^* = \Gamma\Delta\Gamma K = \Gamma\Delta(\Gamma K)$ and hence ΓK^* is the exterior of ΓK which means it is the complement of ΓK in $R_0(M(A))$. Clearly $\Gamma(J \cap K) \subset \Gamma J \cap \Gamma K$ and $\Gamma J \cap \Gamma K = \Gamma(JK) \subset \Gamma(J \cap K)$ and hence $\Gamma(J \cap K) = \Gamma J \cap \Gamma K$. Therefore, Γ is a Boolean homomorphism.

If $\Gamma K = \emptyset$, then K is contained in every prime ideal in $M(A)$ and so $K \subset \Delta M(A) = 0$. Hence Γ is one-to-one. To show that Γ is onto let U be regular open then $\Delta\Gamma(\Delta U)$ is an annihilator ideal such that $\Gamma(\Delta\Gamma(\Delta U)) = U$. Therefore Γ is an isomorphism.

CHAPTER 4

PROJECTIVE RESOLUTIONS

Our main purpose in this chapter is to establish the existence of essentially unique minimal projective resolutions in the category of compact Hausdorff spaces and continuous maps.

<u>Definition</u>: A compact space is called free if it is the Stone-Čech compactification βD of a discrete space D.

<u>Definition</u>: A compact space X is called a projective object if whenever $q : Y \longrightarrow Z$ is onto, any $f : X \longrightarrow Z$ can be lifted, i.e. there is $g : X \longrightarrow Y$ such that $qg = f$.

$$\begin{tikzcd} & X \arrow[dl, dashed, "g"'] \arrow[d, "f"] \\ Y \arrow[r, two heads, "q"'] & Z \end{tikzcd}$$

A number of results are needed before we prove the main existence theorem.

<u>Lemma 1</u>: In Hausdorff spaces, if $t : P \longrightarrow P$ is not the identity then there is a proper closed subspace S such that $S \cup t^{-1}(S) = P$.

Proof: There is a point x such that $t(x) \neq x$. Then x and $t(x)$ have disjoint open neighborhoods U and V. Let $S = P \setminus (U \cap t^{-1}(V))$. If $y \in U \cap t^{-1}(V)$, then $t(y) \in V$ and hence $t(y) \notin U$, so $t(y) \in S$. Therefore $S \cup t^{-1}(S) = P$.

Lemma 2: In compact Hausdorff spaces, if $f : P \longrightarrow X$ is onto, then P has a closed subspace S which is minimal subject to $f(S) = X$.

Proof: Let $\Phi = \{S_\alpha, \alpha \in I \mid f(S_\alpha) = X$ and S_α is closed$\}$. Since $P \in \Phi$, the family Φ is not empty. The family of sets Φ is preordered by inclusion. Let $C = \{C_\alpha, \alpha \in J\}$ be a chain of members of Φ. Since P is compact and C possesses the finite intersection property, $\bigcap C$ is non-empty. $\bigcap C$ is also closed. For a point $x \in X$, the family $\{C_\alpha \cap f^{-1}(x)\}$ of closed subsets of P has the finite intersection property and hence has a non-empty intersection. Hence $(\bigcap C) \cap f^{-1}(x)$ is not empty and $f[\bigcap C]=X$. Now Zorn's Lemma shows the existence of the subspace S.

Lemma 3: In the category of compact Hausdorff spaces, every object is a quotient of a free object and every free object is projective.

Proof: For each object X, let D be a discrete space in one-to-one correspondence with X by $p : D \longrightarrow X$. Since X is compact and p is continuous, p has a unique continuous extension $q : \beta D \longrightarrow X$.

$$\begin{tikzcd}
 & \beta D \arrow[dl, "q"'] \\
X & D \arrow[l, two heads, "p"]
\end{tikzcd}$$

To prove the other part, for any $f : \beta D \longrightarrow Z$ and any onto mapping $h : Y \longrightarrow Z$, define $e : D \longrightarrow Y$ by assigning to each point d of D some point of $h^{-1}(f(d))$. e has a unique continuous extension $g : \beta D \longrightarrow Y$. Since hg and f agree on the dense subset D of βD, they agree everywhere, and βD is projective.

$$\begin{tikzcd}
Z & \beta D \arrow[l, "f"'] \arrow[dl, "g"] \\
Y \arrow[u, two heads, "h"] & D \arrow[l, "e"]
\end{tikzcd}$$

<u>Corollary 1</u>: The projective objects are precisely the retracts of the free objects.

<u>Proof</u>: Suppose P is projective. By the previous Lemma there is an onto mapping $q : \beta D \longrightarrow P$ for some free βD, and the identity

I on P can be lifted to an embedding of P in βD, so that q becomes a retraction.

$$
\begin{tikzcd}
 & \beta D \arrow[d, two heads, "q"] \\
P \arrow[ur, dashed] \arrow[r, two heads, "I"'] & P
\end{tikzcd}
$$

Conversely, if R is a retract of βD by $r : \beta D \longrightarrow R$, then, given $f : R \longrightarrow Z$ and onto mapping $q : Y \longrightarrow Z$, lift fr to $h : \beta D \longrightarrow Y$. We have $qh = fr$, therefore $qh|R = f$. Hence R is projective.

$$
\begin{tikzcd}
 & Y \arrow[ddr, two heads, "q"] & \\
 & \beta D \arrow[u, dashed, "h"] \arrow[dl, "r"] & \\
R \arrow[uur, dashed, "h|R"] \arrow[rr, "f"'] & & Z
\end{tikzcd}
$$

<u>Lemma 4</u>: The following are equivalent:

(i) X is extremally disconnected.

(ii) Disjoint open subsets of X have disjoint closures.

(iii) βX is extremally disconnected.

Proof: (i) $\Longrightarrow$ (ii). Let U and V be disjoint open subsets of X. Then $\mathrm{cl}\,V \cap U = \emptyset$ because U is open. Similarly, $\mathrm{cl}\,V \cap \mathrm{cl}\,U = \emptyset$ because $\mathrm{cl}\,V$ is open.

(ii) $\Longrightarrow$ (iii). Let U be an open subset of βX. Then $U \cap X$ and $X \setminus \mathrm{cl}(U \cap X)$ have disjoint closures whose union is X. Thus, $\mathrm{cl}(U \cap X)$ is clopen in X and therefore, $\mathrm{cl}_{\beta X} U = \mathrm{cl}_{\beta X}(\mathrm{cl}_X(U \cap X))$ is clopen in βX. Hence, βX is extremally disconnected.

(iii) $\Longrightarrow$ (i). Let U be open in X. Then $U = X \cap V$ for some open set V of βX. Because X is dense in βX, $\mathrm{cl}_X U = X \cap \mathrm{cl}_{\beta X} V$, and $\mathrm{cl}_X U$ is thus clopen in X.

Corollary 2: The projective objects in the category of compact Hausdorff spaces are extremally disconnected.

Proof: We only need to show that if P is a retract of a free object βD by $r : \beta D \longrightarrow P$, then P is extremally disconnected. Since D is extremally disconnected, then βD is also extremally disconnected. Let U and V be disjoint open subsets of P. $r^{-1}(U)$ and $r^{-1}(V)$ are disjoint open subsets of βD and hence have disjoint closures in βD. Since

$$\mathrm{cl}_P U \subset \mathrm{cl}_{\beta D}(r^{-1}(U)) \quad \text{and} \quad \mathrm{cl}_P V \subset \mathrm{cl}_{\beta D}(r^{-1}(V)),$$

hence $\mathrm{cl}_P U \cap \mathrm{cl}_P V = \emptyset$ and P is extremally disconnected.

<u>Theorem (Gleason)</u>: For every object X there exists a projective object P and an onto mapping $f : P \longrightarrow X$ such that f maps no proper subobject of P onto X. For any other such P' and $f' : P' \longrightarrow X$ there is an equivalence $e : P \longrightarrow P'$ such that $f = f'e$. Such an object P is called a projective resolution and is denoted by $G(X)$.

<u>Proof</u>: By Lemma 3 let F be free and $g : F \longrightarrow X$ onto. From Lemma 2 we have a minimal subobject P of F such that $g(P) = X$. We will show that P is projective. Let us rename $g|P$ as f. Let i denote the inclusion mapping of P into F. Since F is projective, g can be lifted to P, to $q : F \longrightarrow P$ satisfying $fq = g$. Hence $f q i = g i = f$. If $q i$ were not the identity, Lemma 1 would yield a proper subobject S of P such that $S \cup (qi)^{-1}(S) = P$. Since $qi(qi)^{-1}(S) \subset S$, $f(S)$ contains $fqi(qi)^{-1}(S) = f(qi)^{-1}(S)$. Hence

$$f(S) = f(S) \cup f(qi)^{-1}(S) = f(P) = X,$$

a contradiction. Thus qi is the identity, q is a retraction and P is a retract of F and hence P is projective.

$$\begin{tikzcd} & F \arrow[dl, shift left, "q"] \arrow[d, two heads, "g"] \\ P \arrow[ur, shift left, "i"] \arrow[r, two heads, "g|P=f"'] & X \end{tikzcd}$$

To show that P is unique, let P' be projective, $f':P' \longrightarrow X$ onto, but $f'(S) \neq X$ for all proper subobjects, then we have lifting mappings $e : P \rightarrow P'$ and $d : P' \longrightarrow P$ such that $f'e = f$ and $fd = f'$. So $fde = f$ and using a similar argument to the one used above we show that de is the identity. Likewise ed is also the identity and hence e is an equivalence.

$$\begin{tikzcd} & P \arrow[dl, shift left, "e"] \arrow[d, two heads, "f"] \\ P' \arrow[ur, shift left, "d"] \arrow[r, two heads, "f'"'] & X \end{tikzcd}$$

<u>Lemma 5</u>: Let $f : P \longrightarrow X$ be a map such that $f[S] \neq X$ for any proper closed subset S of P. If A is an open subset of P, then

$$f[A] \subset \mathrm{cl}(X \setminus f[P \setminus A]).$$

<u>Proof</u>: The case when A is empty is trivial. So we may assume that A is non-empty. Let $x \in f[A]$ and let U be a neighborhood of x. We need to show that $U \cap (X \setminus f[P \setminus A]) \neq \phi$. The subset $A \cap f^{-1}[U]$ of P is non-empty because $f^{-1}(x)$ is common to both sets. Since $A \cap f^{-1}[U]$ is open, $P \setminus (A \cap f^{-1}[U])$ is closed, but

$$f[P \setminus (A \cap f^{-1}[U])] \neq X$$

Hence there is a point y in $X \setminus f[P \setminus (A \cap f^{-1}[U])]$. The point y belongs to $X \setminus f[P \setminus A]$ and to $f[f^{-1}[U]] = U$. Therefore $U \cap (X \setminus f[P \setminus A]) \neq \phi$.

<u>Theorem 2 (Gleason)</u>: In the category of compact Hausdorff spaces an object X is projective if and only if it is extremally disconnected.

<u>Proof</u>: Corollary 2 shows that every projective object is extremally disconnected. We still have to show that if an object X is extremally disconnected then it is projective.

Let P be the projective resolution of X. We will show that X is homeomorphic to P and hence X is a projective object. Let $f : P \longrightarrow X$ such that f maps no proper subobject of P onto X. Since P is compact and X is Hausdorff, the mapping f is closed. To show that f is one-to-one we show that distinct points x and y of P have distinct images in X. Let U and V be disjoint neighborhoods of x and y, respectively. Since

$$P = (P \setminus U) \cup (P \setminus V), \qquad \text{we have}$$

$X = f[P] = f[P \setminus U] \cup f[P \setminus V]$. Therefore $X \setminus f[P \setminus U]$ and $X \setminus f[P \setminus V]$ are disjoint open sets in the extremally disconnected space X and hence have disjoint closures. By Lemma 5 $f(x) \in cl(X \;\; f[P \;\; U])$ and $f(y) \in cl(X \;\; f[P \;\; V])$ and so $f(x) \neq f(y)$.

Therefore f is a homeomorphism.

<u>Theorem 3</u>: If X is an infinite discrete space then $G(\beta X \times \beta X)$ is not homeomorphic to $G(\beta X) \times G(\beta X)$.

<u>Proof</u>: βX is extremally disconnected and hence by Theorem 2, $G(\beta X) = \beta X$. Therefore

$$G(\beta X) \times G(\beta X) = \beta X \times \beta X$$

The diagonal $\{(x,x) : x \in X\}$ is a subset of $\beta X \times \beta X$ and is open, but does not have an open closure. Hence $\beta X \times \beta X$ is not extremally disconnected. Since $G(\beta X \times \beta X)$ is extremally disconnected, it cannot be homeomorphic to $G(\beta X) \times G(\beta X)$.

Let X be a compact Hausdorff space. Let $R_0(X)$ be the complete Boolean algebra of all regular open subsets of X. Let $\Omega(X)$ denote the set of all maximal filters F in $R_0(X)$, i.e. $\Omega(X)$ is the Stone space of $R_0(X)$. Since $R_0(X)$ is complete, then by Proposition 2 in Chapter 3 $\Omega(X)$ is extremally disconnected. $\Omega(X)$ is also compact and Hausdorff.

<u>Definition</u>: Let $\lim_X : \Omega(X) \longrightarrow X$ be a map defined by

$$\lim_X(F) = a \qquad \text{where} \qquad a = \lim F.$$

Then $\lim_X$ is onto.

Lemma 6: Let $S = \bigcap_\alpha \Omega_{V_\alpha}$, where Ω_{V_α}, is the clopen set in $\Omega(X)$ determined by the regular open set V_α in X, be a closed set in $\Omega(X)$. Let F be the filter generated by $\{V_\alpha\}$. Then $\lim_X(S) = adh(F)$ = the set of all cluster points of $F = \bigcap_\gamma \overline{F}_\gamma (F_\gamma \in F)$.

Proof: Note that $S = \{G \in \Omega(X) : G \supseteq F\}$. It is clear that $\lim_X(S) \subseteq adh(F)$. Conversely let $x \in adh(F)$. Then for each neighborhood U_x of x, $U_x \cap F_\gamma \neq \phi$, $F_\gamma \in F$. Hence there exists a maximal filter G containing F and all the neighborhoods U_x of x. Clearly $\lim_X G = x$.

Lemma 7: The mapping $\lim_X : \Omega \dashrightarrow X$ is continuous and maps no proper closed subset of Ω onto X.

Proof: To show continuity let $\lim_X(F) = a$, $F \in \Omega(X)$, be an element in X. Let V be an open set containing a. Then $V \in F$. Since X is completely regular, there exists a regular open set $U \subset V$. Then

$$\lim_X(\Omega_U) \subset \mathrm{cl}\, U \subset V.$$

Therefore $\lim_X$ is continuous.

To show the other part, let S be a proper closed subset of Ω and let $G \varepsilon \Omega \setminus S$. Let $S = \bigcap_\alpha \Omega_{V_\alpha}$. Then

$$\begin{aligned} \lim_X(S) &= \mathrm{adh}(F) \qquad \text{where } F \text{ is generated by } \{V_\alpha\} \\ &= \bigcap \mathrm{cl}\, U_\gamma, \quad U_\gamma \varepsilon F \end{aligned}$$

and $G \not\supset F$, i.e. there exists $U \varepsilon F$ and $W \varepsilon G$ such that $U \cap W = \phi$, so $(U \cap V_\alpha) \cap W = \phi$. Then if $a \varepsilon W \subset X$, $a \notin \bigcap \mathrm{cl}\, U_\gamma$. Since $\lim_X(S) = \bigcap \mathrm{cl}\, U_\gamma$, it follows that $\lim_X(S) \neq X$.

For a semisimple ring A with unity the following results were established in Chapter 3.

(i) $J(A) \simeq R_0(M(A))$, where $M(A)$ is the space of maximal ideals of A and $J(A)$ is the complete Boolean algebra of annihilator ideals in A.

(ii) $J(A) \simeq J(Q(A))$.

(iii) If A is rationally complete, $J(A) \simeq A^o$.

<u>Definition</u>: A ring A is called regular if for every $a \varepsilon A$ there exists $x \varepsilon A$ such that $a^2x = a$.

<u>Lemma 8</u>: If A is a regular ring, then $M(A) \simeq M(A^o)$.

Proof: Define $M(A) \longrightarrow M(A^o)$ by $P \rightarrow P \cap A^o$. First we show that this is one-to-one. Let $P \cap A^o = P' \cap A^o$ and let $a \in P$. Suppose $a \notin P'$. There exists $x \in A$ such that $a^2x = a$ because A is regular. Then $a(1 - ax) \in P'$. Since $1 - ax$ is an idempotent, $(1 - ax) \in P' \cap A^o = P \cap A^o$. Thus $1 \in P$; a contradiction. Therefore $P = P'$. Now let $S \in M(A^o)$. Let $P = AS$. P is a proper subset of A, since $1 \notin P$; for this note that $1 \notin S$. Let a_1e_1, $a_2e_2 \in P$. Then clearly $a_1e_1 \cdot a_2e_2 = (a_1a_2)\cdot(e_1e_2) \in P$. Now define $e = e_1 + e_2 - e_1e_2$. Then $e \in S$ and $e_1e = e_1$ and $e_2e = e_2$. Thus $a_1e_1 + a_2e_2 = (a_1e_1 + a_2e_2)e$ is in P. Thus P is an ideal in A. Now we show that P is maximal. To do this, take $a \in A$. Suppose $a \notin P$. There exists $x \in A$ such that $a^2x = a$. Since $a \notin P \supset S$ and $a(1 - ax) \in S$ then $(1 - ax) \in S$. Thus $(1 - ax) \in P$, i.e. P is maximal. Clearly $P \cap A^o = S$.

For each $e \in A^o$, let $M^o(e)$ denote a basic open set $\{S \in M(A^o) : e \notin S\}$ of $M(A^o)$. For each $a \in A$, let $M(a)$ be a basic open set in $M(A)$. Then $M(a)|A^o = M^o(ax)$ for some $x \in A$. Also for each $e \in A^o$, $M^o(e) = M(e)|A^o$. This shows that $P \longrightarrow P \cap A^o$ carries a basic open set onto a basic open set, and so does the inverse mapping. Hence the mapping is a homeomorphism.

Theorem 4: If $M(A)$ is Hausdorff, then $M(Q(A))$ is the Gleason space of $M(A)$.

Proof: Note that $Q(A)$ is regular; thus $M(Q(A)) \simeq M(Q(A)^{\circ})$. Also we have $J(Q(A)) \simeq Q(A)^{\circ} \simeq R_0(M(A))$ since $J(A) \simeq J(Q(A))$ and $J(A) \simeq R_0(M(A))$. Now let $\Omega(M(A))$ be the space of maximal filters F in $R_0(M(A))$. $\Omega(M(A)) \simeq M(R_0(M(A)))$ under the mapping $F \longrightarrow I_F$ where $I_F = \{V : V^* \varepsilon F,\ V^*$ is the complement of $V\}$. Since $M(A)$ is compact and Hausdorff, $\lim_{M(A)} : \Omega \longrightarrow M(A)$ is the projective cover.

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